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On the asymptotic similarity of the zero-pressure-gradient turbulent boundary layer

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We investigate similarity solutions for the outer part of a zero-pressure-gradient turbulent boundary layer in the limit of infinite Reynolds number. Previous work by George (*Phil. Trans. R. Soc.* vol. 365, 2007 p. 789) has suggested that the only appropriate velocity scale for the outer region is U_1, the free-stream velocity. This is based on the fact that scaling with U_1 leads to a mathematically valid similarity solution of the momentum equation for the outer region in the asymptotic limit of infinite Reynolds number. Here we show that the classical scaling using the friction velocity also leads to a valid similarity solution for the outer flow in this limit. Therefore on this basis it is not possible to dismiss the friction velocity as a possible scaling as has been suggested by George (2007) and others. We show that both the free-stream velocity and the friction velocity are potentially valid scalings according to this theoretical criterion.

1. Introduction

The turbulent boundary layer developing in a zero streamwise pressure gradient is a canonical flow case that has been studied for many years. In many respects it represents the simplest possible case for the study of turbulent boundary layers. Of particular interest in this flow is how it develops in the streamwise direction and also the related question of how it behaves at large Reynolds numbers. One reason for the interest is that many practical flows are characterized by very large values of the Reynolds number whereas many carefully controlled laboratory experiments are characterized by much lower Reynolds numbers. A question of some interest, then, is how to extrapolate results measured in laboratory conditions to much higher Reynolds numbers? One way to look at this question is in terms of the correct scaling of the equations. If the correct scaling for the quantities of interest is known, and furthermore is known to apply for all Reynolds numbers, then a sensible extrapolation of results is possible.

This issue of scaling was examined quite early on in turbulence research and it was established by physical arguments and analysis of data that the friction velocity was the appropriate velocity scale for the outer flow. We shall here refer to this as the classical scaling, as described in Clauser (1956), Coles & Hirst (1969), Rotta (1962),

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Perry, Marusic & Jones (2002), and Monkewitz, Chauhan & Nagib (2007). Recently, the arguments leading to this scaling have been questioned by some researchers. One of these alternatives is explored in George & Castillo (1997, henceforth referred to as GC97) in which various criticisms of the classical approach are also discussed. The GC97 theory seeks a similarity solution to the momentum equation for the outer region, in the limit of infinite Reynolds number. This constraint leads them to conclude that using the free-stream velocity, U_1, as the outer velocity scale permits such a similarity solution. Further, GC97 suggest a new principle known as the *Asymptotic Invariance Principle* (AIP) whereby any function representing boundary layer solutions for the outer flow must become independent of the Reynolds number in the limit of $Re \to \infty$. This effectively prescribes that the scalings found in the asymptotic case will also be valid for finite Reynolds numbers. Consequently, GC97 conclude that U_1 is the *correct* velocity scale for the outer region of boundary layers. However, it will be shown below that the classical scaling (with the friction velocity) also satisfies these constraints.

2. The classical scaling

The classical approach to the scaling of the boundary layer has been to consider an inner region close to the wall where the kinematic viscosity, ν, is important but the outer length scale (usually taken as the boundary layer thickness, δ) is unimportant. The effect of the wall shear stress, τ_0, can be expressed in terms of a wall shear velocity $U_\tau = \sqrt{\tau_0/\rho}$ where ρ is the density of the fluid. This approach suggests that in some region close enough to the wall the velocity, U, may be expressed in terms of the wall distance, y, as

$$\frac{U}{U_\tau} = g\left(\frac{yU_\tau}{\nu}\right), \tag{2.1}$$

where g is some unknown but universal function. This relationship may be determined in different ways but its validity (for some small region close to the wall) does not seem to be in dispute. Certainly GC97 accept that this is the appropriate scaling in the inner region.

In the outer part of the flow the usual assumption is that, at sufficiently high Reynolds number (strictly only in the limit as $Re \to \infty$), the viscosity can be neglected and the velocity defect (the difference between the velocity outside the boundary layer, U_1, and the velocity at some point in the layer, U) then depends only on the outer length scale and a velocity scale which is chosen as U_τ again since this is a dynamically important quantity. This may be expressed as

$$\frac{U_1 - U}{U_\tau} = f\left(\frac{y}{\delta}\right), \tag{2.2}$$

where f is some, as yet, unknown function of y/δ. The logarithmic law may be derived in various ways but it does depend on the fact that the velocity scale for both the inner and outer flow is the same and equal to U_τ. The scaling presented above leads to the logarithmic variation of the mean velocity profile in the limit of infinite Reynolds number. This is an important point since it means that if we accept the above scaling and wish to examine the behaviour of various terms in the momentum equation then we can use the logarithmic variation of the mean velocity profile without further assumption.

The essential difference between this classical scaling and the alternative scaling of GC97 is that they take the appropriate velocity scaling for the velocity defect instead to be the free-stream velocity, U_1, i.e. they suggest that the correct defect relation should be

$$\frac{U_1 - U}{U_1} = F\left(\frac{y}{\delta}\right), \tag{2.3}$$

where the upper-case F has been used to emphasize the fact that this is a different function from the classical defect law. If this scaling is accepted then there is no logarithmic variation for the mean velocity profile and other approaches must be used to consider the forms for the mean velocity profile.

3. Outer flow scaling

Here we revisit the analysis of the outer flow region. Following closely the analysis of GC97 we start with the momentum equation for a turbulent boundary layer, which may be written as

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} - \frac{\partial \overline{uv}}{\partial y} + \nu\frac{\partial^2 U}{\partial y^2} - \frac{\partial(\overline{u^2} - \overline{v^2})}{\partial x} \tag{3.1}$$

where x is the streamwise direction, y is the wall-normal direction, U is the mean streamwise velocity, V is the mean velocity normal to the wall, u and v are the fluctuating components of the velocities in the x and y direction respectively, and an overbar denotes a time-averaged quantity. Since we are interested in the zero-pressure-gradient case then the pressure gradient term may be dropped. GC97 note that in the outer region, which is dominated by inertia, the effects of viscosity enter only through the matching to the inner layer. Furthermore, since we are seeking solutions to (3.1) in the limit of infinite Reynolds number the viscous term can safely be neglected from the outset; this is in accordance with the approach of GC97. Hence (3.1) simplifies to

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{\partial \overline{uv}}{\partial y} - \frac{\partial(\overline{u^2} - \overline{v^2})}{\partial x}. \tag{3.2}$$

We are seeking similarity solutions to (3.2) of the form

$$\frac{U_1 - U}{U_0} = f\left(\frac{y}{L}\right), \tag{3.3}$$

$$-\frac{\overline{uv}}{U_{rs}^2} = f_{rs}\left(\frac{y}{L}\right), \tag{3.4}$$

$$\frac{\overline{u^2}}{U_{r1}^2} = f_{r1}\left(\frac{y}{L}\right), \tag{3.5}$$

$$\frac{\overline{v^2}}{U_{r2}^2} = f_{r2}\left(\frac{y}{L}\right), \tag{3.6}$$

where L is some, as yet unspecified, length scale. Similarly there are four unknown velocity scales involved: U_0 for the mean velocity, U_{rs} for the Reynolds shear stress, U_{r1} for the streamwise normal stress and U_{r2} for the wall-normal stress. The functions f, f_{rs}, f_{r1} and f_{r2} are unknown at this stage except in so far as they are only functions of y/L. This is the usual approach to finding similarity solutions of equations. Now the approach is to substitute these expressions into the momentum equation making use of the mean continuity equation to determine an expression for V. This takes some careful algebra, especially since L and all the velocity scales can in principle be functions of the streamwise distance, x. After this substitution the whole equation is made non-dimensional by multiplying through by L/U_0^2. The result after moving everything to one side of the equation and writing $\eta = y/L$ is

$$\frac{LU_1}{U_0^2}\frac{\mathrm{d}U_0}{\mathrm{d}x}f - \frac{U_1}{U_0}\frac{\mathrm{d}L}{\mathrm{d}x}\eta f' - \frac{L}{U_0}\frac{\mathrm{d}U_0}{\mathrm{d}x}(f^2 - I_1 f') + \frac{\mathrm{d}L}{\mathrm{d}x}I_1 f' + \frac{U_{rs}^2}{U_0^2}f'_{rs} + \frac{U_{r1}^2}{U_0^2}\frac{\mathrm{d}L}{\mathrm{d}x}\eta f'_{r1} - \frac{L}{U_0^2}\frac{\mathrm{d}U_{r1}^2}{\mathrm{d}x}f_{r1} - \frac{U_{r2}^2}{U_0^2}\frac{\mathrm{d}L}{\mathrm{d}x}\eta f'_{r2} + \frac{L}{U_0^2}\frac{\mathrm{d}U_{r2}^2}{\mathrm{d}x}f_{r2} = 0 \qquad (3.7)$$

where

$$I_1 = \int_0^\eta f(\eta)\,\mathrm{d}\eta. \qquad (3.8)$$

For self-similar solutions to be permitted all coefficients of functions involving η appearing in (3.7) must scale with each other (or have the same x-dependence, more precisely) which implies

$$\left(a_1 = \frac{LU_1}{U_0^2}\frac{\mathrm{d}U_0}{\mathrm{d}x}\right) \sim \left(a_2 = \frac{U_1}{U_0}\frac{\mathrm{d}L}{\mathrm{d}x}\right) \sim \left(a_3 = \frac{L}{U_0}\frac{\mathrm{d}U_0}{\mathrm{d}x}\right) \sim \left(a_4 = \frac{\mathrm{d}L}{\mathrm{d}x}\right) \sim \left(a_5 = \frac{U_{rs}^2}{U_0^2}\right) \sim \left(a_6 = \frac{U_{r1}^2}{U_0^2}\frac{\mathrm{d}L}{\mathrm{d}x}\right) \sim \left(a_7 = \frac{L}{U_0^2}\frac{\mathrm{d}U_{r1}^2}{\mathrm{d}x}\right) \sim \left(a_8 = \frac{U_{r2}^2}{U_0^2}\frac{\mathrm{d}L}{\mathrm{d}x}\right) \sim \left(a_9 = \frac{L}{U_0^2}\frac{\mathrm{d}U_{r2}^2}{\mathrm{d}x}\right). \qquad (3.9)$$

Note that the case where *all* coefficients are constants (not necessarily zero) is a valid solution to (3.9). Further, one or more of the coefficients can be zero as this is just a special case of being proportional to the others (where the constant of proportionality is zero). We wish to find the appropriate velocity and length scales such that in the limit as $Re \to \infty$ (3.9) holds.

Note that we also have one more condition on the solution–that it satisfies the integral momentum equation for zero pressure gradient.

The obvious length scale for the outer flow is δ, the boundary layer thickness, and this was chosen by GC97 and is also used in the analysis of the classical scaling presented later. The definition for δ used in the following is as defined by Perry *et al.* (2002), but the conclusions that follow are not dependent on the precise definition. For example, the Rotta–Clauser thickness could equally be used.

The solution considered by GC97 is the case where $U_0 = U_1$. This solution coupled with the use of the momentum equation gives the following results:

$$L \sim \delta, \qquad (3.10)$$

$$U_{rs} \sim U_\tau, \qquad (3.11)$$

$$U_{r1} \sim U_{r2} \sim U_1. \qquad (3.12)$$

Note that the last conditions can be found from the terms relating to the streamwise variation of the normal stresses. GC97 omitted these terms from their analysis of the momentum equation but found the same result by considering similarity of the *transport* equations for the normal stresses. However when considering similarity for the transport equation for the Reynolds shear stress GC97 ran into a contradiction which they accepted could be resolved in one way by allowing that …*the term which creates the contradiction must go to zero faster than the other terms so that the offending condition can be removed from the analysis.* (See p. 697 of their paper.)

In terms of the coefficients

$$a_1 = 0, \tag{3.13}$$

$$a_3 = 0, \tag{3.14}$$

$$a_7 = 0, \tag{3.15}$$

$$a_9 = 0. \tag{3.16}$$

From this we see that the choice of U_1 leads to a valid asymptotic solution as was shown by GC97. However, GC97 claim that this is the only possible solution, but this is not so as other possibilities may exist where the coefficients become constants (possibly zero) only in the infinite Reynolds number limit (as opposed to being identically zero as in the case of coefficients a_1, a_3, a_7, a_9 in the above case).

3.1. *The classical solution*

The classical solution suggests that the appropriate velocity scale for the defect law is U_τ (as it is for the inner region) and the length scale is δ (the length is the same for GC97 as in the classical scaling). This corresponds to $U_0 = U_\tau$. The use of U_τ as the velocity scale for the outer region as well as the inner region† leads also to a logarithmic law for the velocity profile in the region of overlap. This definite form for the velocity profile then allows us to examine the behaviour of all the coefficients to see if a similarity solution is possible in the limit of infinite Reynolds number.

The conditions that must be satisfied for self-similarity become

$$\left(a_1 = \frac{\delta U_1}{U_\tau^2}\frac{\mathrm{d}U_\tau}{\mathrm{d}x}\right) \sim \left(a_2 = \frac{U_1}{U_\tau}\frac{\mathrm{d}\delta}{\mathrm{d}x}\right) \sim \left(a_3 = \frac{\delta}{U_\tau}\frac{\mathrm{d}U_\tau}{\mathrm{d}x}\right) \sim \left(a_4 = \frac{\mathrm{d}\delta}{\mathrm{d}x}\right) \sim \left(a_5 = \frac{U_{rs}^2}{U_\tau^2}\right)$$
$$\sim \left(a_6 = \frac{U_{r1}^2}{U_\tau^2}\frac{\mathrm{d}\delta}{\mathrm{d}x}\right) \sim \left(a_7 = \frac{\delta}{U_\tau^2}\frac{\mathrm{d}U_{r1}^2}{\mathrm{d}x}\right) \sim \left(a_8 = \frac{U_{r2}^2}{U_\tau^2}\frac{\mathrm{d}\delta}{\mathrm{d}x}\right) \sim \left(a_9 = \frac{\delta}{U_\tau^2}\frac{\mathrm{d}U_{r2}^2}{\mathrm{d}x}\right). \tag{3.17}$$

Consider first the coefficients that do not contain Reynolds stress velocity scales, and putting $S = U_1/U_\tau$ and noting that for a ZPG layer $(1/U_\tau)(\mathrm{d}U_\tau/\mathrm{d}x) = (-1/S)(\mathrm{d}S/\mathrm{d}x)$, the requirement on these coefficients then becomes

$$\left(a_1 = -\delta\frac{\mathrm{d}S}{\mathrm{d}x}\right) \sim \left(a_2 = S\frac{\mathrm{d}\delta}{\mathrm{d}x}\right) \sim \left(a_3 = -\frac{\delta}{S}\frac{\mathrm{d}S}{\mathrm{d}x}\right) \sim \left(a_4 = \frac{\mathrm{d}\delta}{\mathrm{d}x}\right).$$

In order to determine the behaviour of these coefficients, in the limit, knowledge of the functional form of (3.3) is required. Since in this paper we are interested in testing the compatibility of a similarity solution with the classical scaling we will make use of the Coles (1956) 'law of the wall, law of the wake' given by

$$\frac{U}{U_\tau} = \frac{1}{\kappa}\ln\frac{yU_\tau}{\nu} + A + \frac{\Pi}{\kappa}w\left(\frac{y}{\delta}\right) \tag{3.18}$$

where A is a constant, $w(y/\delta)$ is the Coles (1956) universal wake function and Π represents the 'wake' strength. While Π does depends on the functional form of the wake function it is nevertheless fixed once a form for w is prescribed. Using (3.18)

† The inner scaling is not discussed here since the analysis of GC97 for the inner region is in complete agreement with the classical *inner scaling*.

the velocity defect profile is given by

$$\frac{U_1 - U}{U_\tau} = -\frac{1}{\kappa}\ln\frac{y}{\delta} + \frac{\Pi}{\kappa}\left[w(1) - w\left(\frac{y}{\delta}\right)\right] \tag{3.19}$$

and hence for velocity defect similarity it is required that $\Pi \to$ constant in the limit $Re \to \infty$. Note that in making use of (3.18) we are simply testing the consistency of the classical scaling with a similarity solution.

Evaluating (3.18) at $y = \delta$ gives

$$S = \frac{1}{\kappa}\ln\left(\frac{\delta U_\tau}{\nu}\right) + C, \tag{3.20}$$

$$\delta\frac{dS}{dx} = \frac{S}{(\kappa S + 1)}\frac{d\delta}{dx}, \tag{3.21}$$

where C is a constant. In order to determine the behaviour of $\delta dS/dx$, $S d\delta/dx$ and $d\delta/dx$ in the limit $Re \to \infty$ use is made of the integral momentum equation, given by

$$\frac{d\theta}{dx} = \frac{1}{S^2} \tag{3.22}$$

where θ is the momentum thickness, given by

$$\theta = \int_0^\infty \frac{U}{U_1}\left(1 - \frac{U}{U_1}\right)dy,$$

and using (3.18) this becomes

$$\theta = \delta\left(\frac{C_1}{S} - \frac{C_2}{S^2}\right) \tag{3.23}$$

where

$$C_1 = \int_0^1 \frac{U_1 - U}{U_\tau}d\eta, \tag{3.24}$$

$$C_2 = \int_0^1 \left(\frac{U_1 - U}{U_\tau}\right)^2 d\eta, \tag{3.25}$$

C_1 and C_2 being universal constants for the case of velocity defect similarity. Substituting (3.23) into (3.22) leads to

$$\frac{d\delta}{dx} = \frac{1 - \delta(dS/dx)(2C_2/S - C_1)}{C_1 S - C_2}. \tag{3.26}$$

Combining (3.21) and (3.26) gives

$$a_1 = -\delta\frac{dS}{dx} = \frac{-S}{\kappa C_1 S^2 - \kappa C_2 S + C_2}, \tag{3.27}$$

$$a_2 = S\frac{d\delta}{dx} = \frac{\kappa S^2 + S}{\kappa C_1 S^2 - \kappa C_2 S + C_2}, \tag{3.28}$$

$$a_3 = -\frac{\delta}{S}\frac{dS}{dx} = \frac{-1}{\kappa C_1 S^2 - \kappa C_2 S + C_2}, \tag{3.29}$$

$$a_4 = \frac{d\delta}{dx} = \frac{\kappa S + 1}{\kappa C_1 S^2 - \kappa C_2 S + C_2}. \tag{3.30}$$

Now the limit $Re \to \infty$ corresponds to $\delta U_\tau/\nu \to \infty$ and from (3.20) this also corresponds to $S \to \infty$. Hence the infinite Reynolds number limit for (3.27) to (3.30) can be found by taking $S \to \infty$ and taking this limit gives

$$a_1 = -\delta \frac{\mathrm{d}S}{\mathrm{d}x} \to 0, \quad a_2 = S \frac{\mathrm{d}\delta}{\mathrm{d}x} \to 1/C_1, \quad a_3 = -\frac{\delta}{S} \frac{\mathrm{d}S}{\mathrm{d}x} \to 0, \quad a_4 = \frac{\mathrm{d}\delta}{\mathrm{d}x} \to 0.$$

Note that in the above the limiting form of a_2 and a_4 does not depend on the form of (3.3). However the behaviour of a_1 and a_3 is dependent on the logarithmic law of the wall, law of the wake with the condition $\Pi \to$ constant in the infinite-Re limit. Now consider the coefficients involving Reynolds stress, we have

$$a_5 = \frac{U_{rs}^2}{U_\tau^2}, \quad a_6 = \frac{U_{r1}^2}{U_\tau^2} \frac{\mathrm{d}\delta}{\mathrm{d}x}, \quad a_7 = \frac{\delta}{U_\tau^2} \frac{\mathrm{d}U_{r1}^2}{\mathrm{d}x}, \quad a_8 = \frac{U_{r2}^2}{U_\tau^2} \frac{\mathrm{d}\delta}{\mathrm{d}x}, \quad a_9 = \frac{\delta}{U_\tau^2} \frac{\mathrm{d}U_{r2}^2}{\mathrm{d}x}.$$

Since the coefficient a_2 has been found to be a non-zero constant it follows that at least one of the remaining coefficients must also be a non-zero constant to balance the equation. This is possible if we choose $U_{rs} = U_\tau$ which gives $a_5 = 1$. If we also choose $U_{r1} = U_{r2} = U_\tau$ (which is the usual assumption in the classical method) then in the limit $Re \to \infty$

$$a_6 = a_4 \to 0, \quad a_7 = 2a_3 \to 0, \quad a_8 = a_4 \to 0, \quad a_9 = 2a_3 \to 0,$$

and hence we have shown that the classical solution for the scaling of the zero-pressure-gradient boundary layer satisfies the equations of motion at infinite Reynolds number.

It is worth noting here that at finite Reynolds number the normal stress terms contain Reynolds number corrections as shown by Perry, Henbest & Chong (1986). This means that they cannot be scaled with only δ and U_τ as is done here. The reason that this is a valid procedure is that the finite Reynolds number correction terms are negligible at infinite Reynolds number (see the Appendix) and hence the simple scaling used here is correct in this limit. The manner in which the asymptotic limits are taken (for example, increasing the outer length scale or decreasing the inner length scale) does not change any of outcomes presented here.

It is interesting to note that in this classical solution all terms in the momentum equation but two drop out and the equation reduces to a balance between the terms with the coefficients a_2 and a_5. This amounts to a balance between the Reynolds-shear-stress gradient (in the wall-normal direction) and the change in streamwise momentum.

4. Concluding remarks

In this paper we have shown that the classical scaling of the outer part of the boundary layer leads to an asymptotically valid similarity solution of the momentum equation. Previously, it has been argued (e.g. George 2007) that without this U_τ has no theoretical basis as the appropriate velocity scale for the outer region of boundary layers. Here we have shown that both the free-stream velocity and the friction velocity are potentially valid scalings according to this theoretical criterion. It should be emphasized that using a logarithmic law of the wall to determine the asymptotic behaviour is not an additional assumption in the analysis but follows once the choice of U_τ as the appropriate velocity scale has been made.

The above result also means that classical scaling may also satisfy the *Asymptotic Invariance Principle* of GC97 and therefore (according to this criterion) may also be a potentially valid velocity scale at finite Reynolds numbers. However, satisfying a principle does not imply that the principle itself is in any sense fundamental. One issue that has not been tackled here (or elsewhere it seems) is the question of why any boundary layer should satisfy such a principle. There is no fundamental physical law that states that this should be the case. It is perfectly possible (within the constraints given by the equations of motion) that the various terms in the momentum equations could continue to have different x-dependence as the flow develops (even up to infinite Reynolds number). This type of flow would be called 'non-equilibrium' in the traditional language. It is important to emphasize the point that the AIP is only an hypothesis of George and coworkers. It should also be noted that the fact that the classical scaling does satisfy the principle does not, in itself, suggest that the classical scaling is superior to that of GC97. For this experimental validation is required.

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Appendix. Reynolds normal stresses

Here we investigate the behaviour of the Reynolds normal stress terms appearing in (3.2) when Reynolds-number-dependent terms are included in the functional forms of (3.5) and (3.5). Based on dimensional analysis, in conjunction with the attached eddy hypothesis of Townsend (1976), Perry *et al.* (1986) propose the following similarity laws for the Reynolds normal stresses:

$$\frac{\overline{u^2}}{U_\tau^2} = B_1 - A_1 \ln\left(\frac{y}{\delta}\right) - C'\left(\frac{yU_\tau}{\nu}\right)^{-1/2}, \tag{A 1}$$

$$\frac{\overline{v^2}}{U_\tau^2} = A_2 - \frac{4}{3}C'\left(\frac{yU_\tau}{\nu}\right)^{-1/2}, \tag{A 2}$$

where A_1, A_2, B_1 and C' are constants (note that here we use the notation C' to avoid confusion with the constant C appearing in (3.20)). Equations are valid in the turbulent wall region defined as $\nu/U_\tau \ll y \ll \delta$. Differentiating (A 1) and (A 2) with respect to the streamwise direction yields

$$\frac{\delta}{U_\tau^2}\frac{\partial(\overline{u^2}-\overline{v^2})}{\partial x} = 2(A_2 - B_1)\frac{\delta}{S}\frac{\mathrm{d}S}{\mathrm{d}x} + A_1\left(\frac{\mathrm{d}\delta}{\mathrm{d}x} + \frac{2\delta}{S}\frac{\mathrm{d}S}{\mathrm{d}x}\ln\left(\frac{y}{\delta}\right)\right) - \frac{1}{2}\frac{\delta}{S}\frac{\mathrm{d}S}{\mathrm{d}x}C'\left(\frac{yU_\tau}{\nu}\right)^{-1/2} \tag{A 3}$$

and using (3.29) and (3.30) this can be written as

$$\frac{\delta}{U_\tau^2}\frac{\partial(\overline{u^2}-\overline{v^2})}{\partial x} = -2a_3(A_2 - B_1) + a_4A_1 - 2a_3A_1\ln\left(\frac{y}{\delta}\right) + \frac{1}{2}a_3C'\left(\frac{yU_\tau}{\nu}\right)^{-1/2}. \tag{A 4}$$

Using (3.20) allows (A 4) to be expressed entirely in terms of the similarity variable y/δ and doing this gives

$$\frac{\delta}{U_\tau^2}\frac{\partial(\overline{u^2}-\overline{v^2})}{\partial x} = -2a_3(A_2 - B_1) + a_4A_1 - 2a_3A_1\ln\left(\frac{y}{\delta}\right) + \frac{1}{2}[\exp(\kappa(S-C))]^{-1/2}a_3C'\left(\frac{y}{\delta}\right)^{-1/2}. \tag{A 5}$$

It has been shown previously in the limit $Re \to \infty$ that $a_3 \to 0$ and $a_4 \to 0$. Hence for the forms (A 1) and (A 2) to be compatible with a similarity solution in the limit $Re \to \infty$ we must investigate the behaviour of the coefficient of the $(y/\delta)^{-1/2}$ function.

Writing this coefficient out in full we have

$$a_{10} = \frac{C'}{2}\frac{1}{\sqrt{\exp(\kappa(S-C))}(\kappa C_1 S^2 - \kappa C_2 S + C_2)}.$$

Taking the limit $Re \to \infty$ is equivalent to the limit $S \to \infty$ and it can be shown that for this limit $a_{10} \to 0$ and hence (A 1) and (A 2) are compatible with a similarity solution in the limit $Re \to \infty$.

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